

# Some remarks on groupoids and small categories

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## Abstract

This unpublished note contains some materials taken from my old study note on groupoids and small categories ([1]). It contains a proof for the fact that a groupoid is any group bundle over an equivalence relation. Moreover, the action of a category  $G$  on a category  $H$  as well as the resulting semi-direct product category  $H \times_{\alpha} G$  will be defined (when either  $G$  is a groupoid or  $H^{(0)} = G^{(0)}$ ). If both  $G$  and  $H$  are groupoids, then  $H \times_{\alpha} G$  is also a groupoid. The reason of producing this note is for people who want to check some details in a recent work of Li ([2]).

We consider a small category as a set  $H$  (of morphisms) together with the source and the target maps  $\mathbf{s}, \mathbf{t} : H \rightarrow H^{(0)} \subseteq H$  (i.e., we identify elements in  $H^{(0)}$  with their identity morphisms) as well as the composition  $(h, h') \mapsto hh'$  as a map from  $H^{(2)} := \{(h, h') \in H \times H : \mathbf{s}(h) = \mathbf{t}(h')\}$  to  $H$ . We recall that a groupoid is a small category such that every morphism has an inverse.

**Definition 1** *Let  $X$  be a set and  $R$  be an equivalence relation on  $X$ . Suppose that  $G_{\xi}$  is a group for any  $\xi \in X/R$ . Then  $(X, R, \{G_{\xi}\}_{\xi \in X/R})$  (or simply  $\{G_{\xi}\}_{\xi \in X/R}$ ) is called a group bundle over the equivalence classes of  $R$ .*

Let  $\{G_{\xi}\}_{\xi \in X/R}$  be a group bundle over the equivalence classes of an equivalence relation  $R$  on  $X$  and let

$$G := \{(x, g, y) : \xi \in X/R; x, y \in \xi; g \in G_{\xi}\}$$

and  $\mathbf{t}, \mathbf{s} : G \rightarrow X$  are defined by  $\mathbf{t}(x, g, y) = x$  and  $\mathbf{s}(x, g, y) = y$ . Moreover, we have  $i : X \rightarrow G$  given by  $i(x) = (x, e, x)$ . We can define the product and inverse as follows:  $(x, g, y)(y, h, z) = (x, gh, z)$  and  $(x, g, y)^{-1} = (y, g^{-1}, x)$ . It is not hard to check that  $G$  is a groupoid. Groupoids that defined by group bundles in this way are said to be of *standard type*. The interesting thing is that every groupoid is actually of standard type.

**Proposition 2** *There is a natural one to one correspondence between groupoids with unit space  $X$  and group bundles over the equivalence classes of equivalence relations on  $X$ .*

**Proof:** We have already seen in the above how we can construct a groupoid with unit space  $X$  from a group bundle over the equivalence classes of an equivalence relation on  $X$  and this gives a correspondence  $\mathcal{C}$ . We give the inverse construction in the following. Let  $(H, X, \mathbf{t}, \mathbf{s}, i)$  be a groupoid. For any  $x, y \in X$ , we let

$${}_x H_y := \mathbf{t}^{-1}(x) \cap \mathbf{s}^{-1}(y).$$

It is clear that  ${}_x H_x$  is a group. Moreover, if we take any  $g \in {}_x H_y$ , the map  $h \mapsto g^{-1}hg$  is an isomorphism from  ${}_x H_x$  to  ${}_y H_y$ . Furthermore, we have  ${}_x H_x g = {}_x H_y = g {}_y H_y$ . We define an equivalence relation  $S$  on  $X$  by  $(x, y) \in S$  if  ${}_x H_y \neq \emptyset$ . For any  $\xi \in X/S$ , we fix a representative  $x_{\xi} \in \xi$ . Then, we obtain an assignment of groups  $\xi \mapsto {}_{x_{\xi}} H_{x_{\xi}}$ . It is clear that if the groupoid  $H$  is defined by a group bundle  $\{G_{\xi}\}_{\xi \in X/R}$ , then  $S = R$  and  ${}_{x_{\xi}} H_{x_{\xi}} \cong G_{\xi}$  for any  $\xi \in X/R$ . Therefore, the correspondence  $\mathcal{C}$  is injective.

On the other hand, we need to show that  $\mathcal{C}$  is surjective. It is required to show that starting from a groupoid  $H$  with unit space  $X$ , the canonical groupoid  $G$  associate with the group bundle  $\{_{x \in X} H_x\}$  over the equivalence classes of the equivalence relation  $S$  is isomorphic to  $H$ . In fact, for any  $\xi \in X/R$  and any  $x \in \xi$ , we fix an element  $l_x \in {}_x H_x$ . For any  $g \in H$ , let  $x = \mathbf{t}(g)$  and  $y = \mathbf{s}(g)$ . Since  $(x, y) \in S$ , they belong to an equivalence class  $\xi$ . Now, we define

$$\Phi(g) := (x, l_x \cdot g \cdot l_y^{-1}, y) \in G.$$

It is clear that  $\Phi$  is a groupoid homomorphism. Moreover,  $\Phi$  is injective because  $l_x \cdot g \cdot l_y^{-1} = l_x \cdot g' \cdot l_y^{-1}$  will imply that  $g = g'$  and  $\Phi$  is surjective since  $\Phi(l_x^{-1} \cdot h \cdot l_y) = (x, h, y)$  (for any  $(x, h, y) \in G$ ).  $\square$

In the case when  $H$  is only a small category, one can also define the equivalence relation  $R$  as well as  ${}_x H_y$  as in the above.

**Corollary 3** *Let  $\alpha$  and  $\beta$  be actions of groups  $G$  and  $H$  on sets  $X$  and  $Y$  respectively. Then  $X \times_\alpha G \cong Y \times_\beta H$  as groupoids if and only if there exists a bijection  $\psi : X \rightarrow Y$  such that the restriction of  $\psi$  on every orbit  $O$  of  $\alpha$  is a bijection onto an orbit of  $\beta$  whose stabilization group (a subgroup of  $H$ ) is isomorphic to that of  $O$  (a subgroup of  $G$ ).*

**Definition 4** *Let  $G$  and  $H$  be small categories. Suppose that  $\varphi : H^{(0)} \rightarrow G^{(0)}$ . We let*

$$G \times^\varphi H := \{(g, h) \in G \times H : \mathbf{s}(g) = \varphi(\mathbf{t}(h)) = \varphi(\mathbf{s}(h))\}.$$

A left action of  $G$  on  $H$  with respect to  $\varphi$  is a map  $(g, h) \mapsto \alpha_g(h)$  from  $G \times^\varphi H$  to  $H$  such that for any  $(g', g) \in G^{(2)}$ ,  $(h', h) \in H^{(2)}$  and  $u \in H^{(0)}$  with  $(g, h), (g, u), (g, h') \in G \times^\varphi H$ , we have:

- (0).  $\mathbf{s}(\alpha_g(u)) = \mathbf{t}(\alpha_g(u))$ ;
- (I).  $\mathbf{s}(\alpha_g(\mathbf{s}(h))) = \mathbf{s}(\alpha_g(h))$  ;
- (II).  $\mathbf{t}(\alpha_g(\mathbf{t}(h))) = \mathbf{t}(\alpha_g(h))$ ;
- (III).  $\varphi(\mathbf{s}(\alpha_g(u))) = \mathbf{t}(g)$ ;
- (IV).  $\alpha_{\varphi(\mathbf{t}(h))}(h) = h$ ;
- (V).  $\alpha_{g'}(\alpha_g(h)) = \alpha_{g'g}(h)$ ;
- (VI).  $\alpha_g(h'h) = \alpha_g(h')\alpha_g(h)$ .

For simplicity, we say that  $(\varphi, \alpha)$  (or simply  $\alpha$  if  $\varphi$  is understood) is a left action of  $G$  on  $H$ .

**Remark 5** (a) From (I), (II) and (III), we know that

$$\varphi(\mathbf{s}(\alpha_g(h))) = \mathbf{t}(g) = \varphi(\mathbf{t}(\alpha_g(h))). \quad (1)$$

Therefore,  $\varphi(\mathbf{t}(\alpha_g(h))) = \mathbf{t}(g) = \mathbf{s}(g')$  and  $\varphi(\mathbf{s}(\alpha_g(h))) = \mathbf{s}(g')$  which show that the left hand side of (V) makes sense. The right hand side of (V) clearly makes sense because  $\mathbf{s}(g'g) = \mathbf{s}(g)$ .

(b) It is clear that the left hand side of (VI) makes sense as  $\mathbf{t}(h'h) = \mathbf{t}(h')$  and  $\mathbf{s}(h'h) = \mathbf{s}(h)$ . On the other hand, since  $\mathbf{t}(\alpha_g(h)) = \mathbf{t}(\alpha_g(\mathbf{t}(h))) = \mathbf{t}(\alpha_g(\mathbf{s}(h'))) = \mathbf{s}(\alpha_g(\mathbf{s}(h'))) = \mathbf{s}(\alpha_g(h'))$  (because of (0), (I) and (II)), the right hand side of (VI) makes sense.

(c) For any  $y \in G^{(0)}$ , we denote  $X_y = \varphi^{-1}(y)$ . For any  $g \in G$ , the left action  $\alpha$  defines a natural transform

$$\alpha_g : H|_{X_{\mathbf{s}(g)}} \rightarrow H|_{X_{\mathbf{t}(g)}}$$

(where  $H|_Z$  is the full subgroupoid of  $H$  with  $H|_Z^{(0)} = Z$ ). If  $y \in G^{(0)}$ , then  $\alpha_y$  is the identity map from  $H|_{X_y}$  to  $H|_{X_y}$ . Moreover, if  $G$  is a groupoid, then  $\alpha_g$  is an isomorphism.

**Lemma 6** Suppose that  $G$  is a category acting on another category  $H$  through a left action  $(\varphi, \alpha)$ .

(a) If  $G_\varphi$  is the full subcategory of  $G$  with  $G_\varphi^{(0)} = \varphi(H^{(0)})$ , then

$$G \times^\varphi H = G_\varphi \times^\varphi H.$$

Consequently, we can always assume that  $\varphi$  is surjective.

(b) If  $G$  is a groupoid and  $(g, u) \in G \times^\varphi H$  with  $u \in H^{(0)}$ , then  $\alpha_g(u) \in H^{(0)}$ . In this case,  $\alpha$  induces an action  $\alpha^{(0)}$  of  $G$  on  $H^{(0)}$ .

(c) Suppose that both  $G$  and  $H$  are groupoids and  $(g, h) \in G \times^\varphi H$ . Then

$$\alpha_g(h)^{-1} = \alpha_g(h^{-1}). \quad (2)$$

**Proof:** (a) For any  $(g, h) \in G \times^\varphi H$ , we have  $\mathbf{s}(g) = \varphi(\mathbf{s}(h)) \in G_\varphi^{(0)}$  and  $\mathbf{t}(g) = \varphi(\mathbf{t}(\alpha_g(h))) \in G_\varphi^{(0)}$  (by Remark 5(a)) and the equality follows.

(b) If  $v = \mathbf{s}(\alpha_g(u)) = \mathbf{t}(\alpha_g(u))$ , then

$$\alpha_{g^{-1}}(\alpha_g(u)) = \alpha_{g^{-1}}(\alpha_g(u)v) = \alpha_{\mathbf{s}(g)}(u)\alpha_{g^{-1}}(v) = u\alpha_{g^{-1}}(v) = \alpha_{g^{-1}}(v)$$

and so  $\alpha_g(u) = v$ .

(c) It is clear that  $(g, h^{-1}) \in G \times^\varphi H$ . We have by (II) and (IV),

$$\alpha_g(h)\alpha_g(h^{-1}) = \alpha_g(\mathbf{t}(h)) = \mathbf{t}(\alpha_g(h))$$

and similarly,  $\alpha_g(h^{-1})\alpha_g(h) = \mathbf{s}(\alpha_g(h))$ . □

**Remark 7** Suppose that  $G$  is a groupoid. By Lemma 6(b), one can replace Conditions (0) - (III) with the following three conditions:

(I').  $\alpha_g(\mathbf{s}(h)) = \mathbf{s}(\alpha_g(h))$ ;

(II').  $\alpha_g(\mathbf{t}(h)) = \mathbf{t}(\alpha_g(h))$ ;

(III').  $\varphi(\alpha_g(u)) = \mathbf{t}(g)$ .

Note that condition (I') implies that  $\alpha_g(H^{(0)}) \subseteq H^{(0)}$  for every  $g \in G$ .

**Proposition 8** Suppose that  $G$  is a groupoid acting on a small category  $H$  by a left action  $(\varphi, \alpha)$  and

$$H \times_\alpha G := \{(h, g) \in H \times G : \mathbf{t}(g) = \varphi(\mathbf{s}(h)) = \varphi(\mathbf{t}(h))\}.$$

For any  $(h, g) \in H \times_\alpha G$ , we set

$$\mathbf{s}(h, g) := \alpha_{g^{-1}}(\mathbf{s}(h)) \quad \text{and} \quad \mathbf{t}(h, g) := \mathbf{t}(h)$$

(here, we identify  $u \in H^{(0)}$  with its canonical image  $(u, \varphi(u)) \in H \times_\alpha G$ ). Moreover, if  $(h, g), (h', g') \in H \times_\alpha G$  satisfying  $\mathbf{s}(\alpha_{g^{-1}}(h)) = \mathbf{t}(h')$ , we define

$$(h, g)(h', g') := (h\alpha_g(h'), gg').$$

This turns  $H \times_\alpha G$  into a small category. If, in addition,  $H$  is a groupoid, then  $H \times_\alpha G$  is also a groupoid with

$$(h, g)^{-1} = (\alpha_{g^{-1}}(h^{-1}), g^{-1}).$$

**Proof:** Since

$$\mathbf{t}(g') = \varphi(\mathbf{t}(h')) = \varphi(\mathbf{s}(\alpha_{g^{-1}}(h))) = \mathbf{t}(g^{-1}) = \mathbf{s}(g) \quad (3)$$

(by Equality (1)), the product  $gg'$  is valid. Secondly, as  $(h', g') \in H \times_\alpha G$ , Equation (3) shows that  $(g, h') \in G \times^\varphi H$ . Furthermore,

$$\mathbf{t}(\alpha_g(h')) = \alpha_g(\mathbf{t}(h')) = \alpha_g(\mathbf{s}(\alpha_{g^{-1}}(h))) = \mathbf{s}(h) \quad (4)$$

by the hypothesis as well as (I'), (II') & (IV) and so  $(h, \alpha_g(h')) \in H^{(2)}$ . Therefore, the product is well defined. It is not hard to see that the product is associative. Now, suppose that  $H$  is a groupoid. Then clearly  $(g^{-1}, h^{-1}) \in G \times^\varphi H$ , and by Equation (1),

$$\varphi(\mathbf{s}(\alpha_{g^{-1}}(h^{-1}))) = \mathbf{t}(g^{-1}) = \varphi(\mathbf{t}(\alpha_{g^{-1}}(h^{-1}))).$$

Thus,  $(\alpha_{g^{-1}}(h^{-1}), g^{-1}) \in H \times_\alpha G$ . Moreover, by (I') and (V),

$$\mathbf{s}(\alpha_{g^{-1}}(h^{-1}), g^{-1}) = \alpha_g(\mathbf{s}(\alpha_{g^{-1}}(h^{-1}))) = \mathbf{t}(h) = \mathbf{t}(h, g)$$

and

$$\mathbf{t}(\alpha_{g^{-1}}(h^{-1}), g^{-1}) = \mathbf{t}(\alpha_{g^{-1}}(h^{-1})) = \alpha_{g^{-1}}(\mathbf{t}(h^{-1})) = \mathbf{s}(h, g).$$

Now, it is easy to check that  $(\alpha_{g^{-1}}(h^{-1}), g^{-1})$  is the inverse of  $(h, g)$ .  $\square$

**Remark 9** (a) Suppose that  $\bar{H} = \{\bar{h} : h \in H\}$  is the opposite category of  $H$  (i.e.  $\mathbf{s}(\bar{h}) = \mathbf{t}(h)$ ,  $\mathbf{t}(\bar{h}) = \mathbf{s}(h)$ ,  $\bar{H}^{(2)} = \{(\bar{h}, \bar{h}') : (h', h) \in H^{(2)}\}$  and  $\bar{h}\bar{h}' = \bar{h}'\bar{h}$ ). Since  $\bar{H}^{(0)} = H^{(0)}$ , it is not hard to check that a left action  $(\varphi, \alpha)$  of  $G$  on  $H$  induces a left action  $(\bar{\alpha}, \varphi)$  on  $\bar{H}$  given by  $\bar{\alpha}_g(\bar{h}) = \overline{\alpha_g(h)}$ .

(b) One can also turn  $G \times^\varphi H$  into a small category with the following source map, target map and product:

$$\mathbf{s}(g, h) := \mathbf{s}(h), \quad \mathbf{t}(g, h) := \mathbf{t}(\alpha_g(h)) \quad \text{and} \quad (g', h')(g, h) := (g'g, \alpha_{g^{-1}}(h')h)$$

(if  $\mathbf{s}(h') = \mathbf{t}(\alpha_g(h))$ ). If  $\bar{H}$  and  $\bar{\alpha}$  is as in part (a), then it is not hard to check that the map  $\Psi : \bar{H} \times_{\bar{\alpha}} G \rightarrow G \times^\varphi H$  given by  $\Psi(\bar{h}, g) = (g^{-1}, h)$  is an isomorphism of the two categories. Consequently, if  $H$  is a groupoid, the map  $\Psi : H \times_\alpha G \rightarrow G \times^\varphi H$  defined by  $\Psi(h, g) = (g^{-1}, h^{-1})$  is a groupoid isomorphism.

We require that  $G$  is a groupoid in Proposition 8 because we need to define  $\mathbf{s}(h, g)$ . There is a situation when  $\mathbf{s}(h, g)$  can be determined without the existence of  $g^{-1}$ . Note that since  $G$  is groupoid,  $\varphi(\alpha_{g^{-1}}(\mathbf{s}(h))) = \mathbf{t}(g^{-1}) = \mathbf{s}(g)$  (by (III')) and one has  $\alpha_{g^{-1}}(\mathbf{s}(h)) = \varphi^{-1}(\mathbf{s}(g))$  if  $\varphi$  is injective. However, in this case, one can even assume that  $\varphi$  is actually bijective (because of Lemma 6(a)), and one can identify  $H^{(0)}$  with  $G^{(0)}$ .

The following proposition follows from nearly the same argument as that of Proposition 8 except that we need to replace equalities (4) by the following:

$$\mathbf{t}(\alpha_g(h')) = \mathbf{t}(\alpha_g(\mathbf{t}(h'))) = \mathbf{s}(\alpha_g(\mathbf{t}(h'))) = \mathbf{t}(g) = \mathbf{s}(h).$$

**Proposition 10** Let  $G$  and  $H$  be small categories such that  $H^{(0)} = G^{(0)}$ . Suppose that  $(\text{id}, \alpha)$  a left action of  $G$  on  $H$ . For any  $(h, g) \in H \times_\alpha G := \{(h, g) \in H \times G : \mathbf{t}(g) = \mathbf{s}(h) = \mathbf{t}(h)\}$ , we set

$$\mathbf{s}(h, g) := (\mathbf{s}(g), \mathbf{s}(g)) \quad \text{and} \quad \mathbf{t}(h, g) := (\mathbf{t}(h), \mathbf{t}(h)).$$

Moreover, if  $(h, g), (h', g') \in H \times_\alpha G$  satisfying  $\mathbf{s}(g) = \mathbf{t}(h')$  we define

$$(h, g)(h', g') := (h\alpha_g(h'), gg').$$

This turns  $H \times_\alpha G$  into a small category.

The category  $H \times_\alpha G$  is called the *semi-direct product of  $H$  and  $G$  under the left action  $(\varphi, \alpha)$*  (when  $G$  is a groupoid or when  $H^{(0)} = G^{(0)}$ ).

Now, let  $G$  be a category acting on a category  $H$  through a left action  $(\varphi, \alpha)$  and  $R$  be the canonical equivalence relation on  $G^{(0)}$  defined by  $G$ . If  $G^{(0)} = H^{(0)}$  and  $\varphi = \text{id}$ , then  ${}_v(H \times_\alpha G)_u = {}_vH_v \times {}_vG_u$  and the composition is given by the map  ${}_vG_u \times {}_uH_u \rightarrow {}_vH_v$  ( $u, v \in H^{(0)}$ ). Thus, in order to construct  $H \times_\alpha G$ , one needs only to know the semi-group bundles  ${}_uH_u$  ( $u \in H^{(0)}$ ). One can also use the idea from this decomposition to construct a “restricted semi-direct product”.

For any  $u, v \in H^{(0)}$  with  $\varphi(u)R\varphi(v)$ , we define

$${}_u\tilde{G}_v := \{g \in {}_{\varphi(u)}G_{\varphi(v)} : u = \alpha_g(v)\}$$

(which could be empty). For any  $g \in {}_u\tilde{G}_v$  and  $h \in {}_v\tilde{G}_w$ , we have  $gh \in {}_{\varphi(u)}G_{\varphi(w)}$  and  $\alpha_{gh}(w) = \alpha_g(\alpha_h(w)) = u$  and thus,  $gh \in {}_u\tilde{G}_w$ .

Let  $\tilde{G}$  be category with object  $H^{(0)}$  and with morphisms from  $u \in H^{(0)}$  to  $v \in H^{(0)}$  being  ${}_u\tilde{G}_v$  (note that the identity morphism in  ${}_u\tilde{G}_u$  is  $\varphi(u) \in {}_u\tilde{G}_u$ ).

Consider  $j$  be the canonical natural transform from  $\tilde{G}$  to  $G$  that send an object  $u$  to  $\varphi(u)$  and send a morphism  $g \in {}_u\tilde{G}_v$  to  $g \in {}_{\varphi(u)}G_{\varphi(v)}$ . Therefore, for any  $g \in {}_u\tilde{G}_v$ , we have  $\mathbf{s}(j(g)) = \varphi(\tilde{\mathbf{s}}(g))$  and  $\mathbf{t}(j(g)) = \varphi(\tilde{\mathbf{t}}(g))$  (where  $\tilde{\mathbf{s}}$  and  $\tilde{\mathbf{t}}$  are the source and the target maps in  $\tilde{G}$ ). It is clear that for any  $(g, h) \in \tilde{G} \times^{\text{id}} H$ , we have  $(j(g), h) \in G \times^\varphi H$  and we can define  $\tilde{\alpha}_g(h) = \alpha_{j(g)}(h)$ . It is not hard to see that  $(\text{id}, \tilde{\alpha})$  is an action of  $\tilde{G}$  on  $H$ . By Proposition 10, one can define  $H \times_{\tilde{\alpha}} \tilde{G}$  which is called the *restricted semi-direct product of  $H$  and  $G$  under the action  $\alpha$* .

**Remark 11** (a) If  $G$  is a groupoid, then for any  $g \in {}_u\tilde{G}_v$ , we also have  $g^{-1} \in {}_{\varphi(v)}G_{\varphi(u)}$  and  $v = \alpha_{g^{-1}}(u)$  which shows that  $g^{-1} \in {}_v\tilde{G}_u$  and so,  $\tilde{G}$  is also a groupoid. Note that  $(h, j(g)) \in H \times_\alpha G$  if  $(h, g) \in H \times_{\tilde{\alpha}} \tilde{G}$  and the map  $\Psi$  that send  $(h, g)$  to  $(h, j(g))$  is an injective natural transform that is an identity on the space of object  $H^{(0)}$ . In general,  $\Psi$  is not surjective because for any  $(h, g) \in H \times_{\tilde{\alpha}} \tilde{G}$ , one has  $\mathbf{s}(h) = \mathbf{t}(h)$ . That is why  $H \times_{\tilde{\alpha}} \tilde{G}$  is called the restricted semi-direct product. Note that if  $G$  and  $H$  are groups, then  $H \times_{\tilde{\alpha}} \tilde{G} = H \times_\alpha G$ .

(b) Suppose that  $G$  is a groupoid and  $H'$  is the semi-group bundle of  $H$  (i.e.  ${}_uH'_u = {}_uH_u$  and  ${}_uH'_v = \emptyset$  if  $u \neq v$ ). Then  $(\varphi, \alpha)$  induces an action  $(\varphi, \alpha')$  of  $G$  on  $H'$ . Obviously,  $H' \times_{\alpha'} G$  is a subcategory of  $H \times_\alpha G$ . Moreover, it is clear that  $H' \times_{\alpha'} G$  contains the image of the natural transform  $\Psi$  in part (a). Conversely, for any  $(h, g) \in H' \times_{\alpha'} G$ , we see that  $\mathbf{t}(g) = \varphi(\mathbf{s}(h))$  and  $\mathbf{s}(h) = \mathbf{t}(h)$ . We let  $u = \mathbf{s}(h)$  and  $v = \alpha_{g^{-1}}(u)$ . Then  $\varphi(v) = \mathbf{s}(g)$  and so  $g \in {}_u\tilde{G}_v$ . Thus,  $\Psi(H \times_{\tilde{\alpha}} \tilde{G}) = H' \times_{\alpha'} G$ . Consequently, if  $H$  is a semi-group bundle, then  $H \times_\alpha G \cong H \times_{\tilde{\alpha}} \tilde{G}$  and this applies, in particular, to the case when  $H$  is a set. Now, if  $G$  is only a category and  $H$  is a semi-group bundle, then one can also define  $H \times_\alpha G$  as  $H \times_{\tilde{\alpha}} \tilde{G}$ .

**Example 12** (a) Suppose that  $G$  is a group and  $H$  is the groupoid given by an equivalence relation  $\sim$  on  $X = H^{(0)}$ . Then a left action of  $G$  on  $H$  is the same as an action  $\beta$  of  $G$  on  $X$  that respects  $\sim$  in the following sense: for any  $u, v \in X$  and  $g \in G$ ,

$$u \sim v \quad \text{implies} \quad \beta_g(u) \sim \beta_g(v).$$

(b) If both  $G$  and  $H$  are group (i.e. both  $G^{(0)}$  and  $H^{(0)}$  are singletons), then action of  $G$  on  $H$  in the above coincides with the usual definition of action of  $G$  on  $H$  by automorphisms and  $H \times_\alpha G$  is the usual semi-direct product.

(c) If  $G$  is a groupoid and  $H$  is a set  $X$  together with the trivial groupoid structure (i.e.  $H^{(2)} = \{(x, x) : x \in X\}$ ), then action of  $G$  on  $H$  in the above coincides with the usual definition of action of  $G$  on the set  $X$  and  $X \times_\alpha G$  is the usual semi-direct product of  $G$  and  $X$ .

(d) If  $G$  is a groupoid and  $G = G^{(0)}$ , then  $H \times_\alpha G \cong \{h \in H : \varphi(s(h)) = \varphi(t(h))\}$  as categories. In particular, if  $\varphi(H^{(0)})$  is a singleton, then  $H \times_\alpha G \cong H$ . On the other hand, if  $\varphi$  is injective, then  $H \times_\alpha G$  coincides with the semi-group bundle  $\{h \in H : s(h) = t(h)\}$ .

(e) Let  $G$  be a groupoid,  $H = G$  and  $\varphi$  be the identity map. We define an action of  $G$  on itself by inner automorphism, i.e.  $\alpha_g(h) = ghg^{-1}$  (for any  $(g, h) \in G \times^{\text{id}} G$ ). It is not hard to check that the conditions in Definition 4 are satisfied. Moreover, as in the case of group, there is a canonical groupoid homomorphism  $\Psi$  from  $G \times_\alpha G$  to  $G$  that sends  $(h, g)$  to  $hg^{-1}$ . However, unlike the group case, the “kernel of  $\Psi$ ” equals

$$\{(h, g) \in G \times_\alpha G : \Psi(h, g) \in G^{(0)}\} = \{(g, g^{-1}) \in G \times G : s(g) = t(g)\} \cong \{g \in G : s(g) = t(g)\}$$

and is not isomorphic to the whole of  $G$  (unless  $G^{(0)}$  is a singleton).

(f) Let  $X$  be a set of Hilbert spaces,  $H$  is the category of bounded linear maps between elements in  $X$  and  $G$  is the category of isometries from one element of  $X$  to another one. We define a left action  $\alpha$  of  $G$  on  $H$  as follows. If  $\Psi : K_1 \rightarrow K_2$  is an isometry and  $T : K_1 \rightarrow K_1$  is a bounded linear map, then  $\alpha_\Psi(T) = \Psi \circ T \circ \Psi^*$ . It is not hard to see that  $(\text{id}, \alpha)$  satisfy the conditions in Definition 4. For any  $(T, \Psi) \in H \times_\alpha G$ , we let  $\Phi(T, \Psi) = T \circ \Psi^*$ . Then it is not hard to see that  $\Phi$  is a natural transform from  $H \times_\alpha G$  to  $H$ . In general,  $\Phi$  is surjective but not injective.

## References

- [1] C.K. Ng, A note on groupoids, unpublished study note.
- [2] H. Li,  $C^*$ -crossed product of groupoid actions on categories, in preparation.

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